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POSITIVE TRANSFORMATIONS ON GROUP SPACES. I [1]

By

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1. Introduction

Let $\mathcal{G}$ be a topological group and $p_t(E)$ a semi-group of finite positive measures on $\mathcal{G}$. That is to say,

$\int_{\mathcal{G}} p_s(d\sigma)p_t(\sigma^{-1}E) = p_{s+t}(E)$ $s,t > 0$ and $E \subset \mathcal{G}$. It follows that $p_s(\mathcal{G})p_t(\mathcal{G}) = p_{s+t}(\mathcal{G})$, so that $p_t(\mathcal{G}) = e^{ct}$ for some c; after replacing $p_t(E)$ by $e^{-ct}p_t(E)$ we may therefore assume $p_t(\mathcal{G}) = 1$ for $t > 0$. We shall also assume that $\lim_{t \searrow 0} p_t(E) = 1$ whenever E is a neighborhood of the identity e of $\mathcal{G}$.

The $p_t(E)$ give rise to a semi-group S_t of linear transformations on the Banach space of bounded continuous functions on $\mathcal{G}$:

$$S_t f.(\tau) = \int_{\mathcal{G}} p_t(d\sigma)f(\tau\sigma),$$

where $S_t f.(\tau)$ is the value of the function $S_t f$ at τ. The S_t have these properties:

$$\lim_{t \searrow 0} S_t f = f$$

$$\inf_{\tau \varepsilon \mathcal{G}} S_t f.(\tau) \geqq \inf_{\tau \varepsilon \mathcal{G}} f(\tau) \qquad t > 0$$

$$L_\sigma S_t = S_t L_\sigma \qquad \sigma \varepsilon \mathcal{G},\ t > 0$$

Here L_σ is the left translation $L_\sigma f.(\tau) = f(\sigma^{-1}\tau)$ and the first equality holds for every f which is uniformly continuous in the left uniform topology of $\mathcal{G}$. The second property mentioned is equivalent to the statement that $S_t f = f$ if f is a constant and $S_t f \geqq 0$ if $f \geqq 0$.

If we suppose that $\mathcal{G}$ is locally compact we may, altering the situation slightly, start with a semi-group S_t and derive a semi-group $p_t(E)$ from which the S_t are obtained in the manner just described. If $\mathcal{G}$ is not compact, let $\bar{\mathcal{G}}$ be its one-point compactification; the point at infinity a is to be kept fixed under automorphisms and left and right translations of $\mathcal{G}$. When $\mathcal{G}$ is compact take $\bar{\mathcal{G}}$ to be $\mathcal{G}$ itself; all statements concerning a are vacuous in this case. Now let $\bar{\mathcal{C}}$ be the Banach space of continuous functions on $\bar{\mathcal{G}}$ and $(S_t)_{t>0}$ a semi-group of bounded linear transformations on $\bar{\mathcal{C}}$ satisfying

(1) $\lim_{t\searrow 0} S_t f = f \qquad f \in \bar{\mathcal{C}}$

(2) $\min_{\tau\in\bar{\mathcal{G}}} S_t f.(\tau) \geqq \min_{\tau\in\bar{\mathcal{G}}} f(\tau) \qquad f \in \bar{\mathcal{C}},\ t > 0$

(3) $L_\sigma S_t = S_t L_\sigma \qquad t > 0,\ \sigma \in \mathcal{G}$

It follows from (2) that the bound of S_t is 1. Hence for each $t > 0$ and each τ in $\bar{\mathcal{G}}$ there is a measure $q_t(\tau,E)$ on $\bar{\mathcal{G}}$ such that

$$S_t f.(\tau) = \int_{\bar{\mathcal{G}}} q_t(\tau,d\sigma)f(\sigma).$$

It also follows from (2) that $q_t(\tau,E) \geqq 0$ and $q_t(\tau,\bar{\mathcal{G}}) = 1$. Now (3) implies that $q_t(\sigma\tau,\sigma E) = q_t(\tau,E)$ for $\sigma\varepsilon\mathcal{G}$, $\tau\varepsilon\bar{\mathcal{G}}$, $E \subset \bar{\mathcal{G}}$. Thus there is a measure $p_t(E) = q_t(e,E)$ on $\bar{\mathcal{G}}$ such that $q_t(\tau,E) = p_t(\tau^{-1}E)$ for every τ in $\mathcal{G}$. It is easy to see that if $\mathcal{G}$ is not compact $q_t(a,E)$ is 1 or 0 according as E contains a or not. Otherwise $q_t(a,E)$ would be strictly positive for some compact set E not containing a; we could then take a sequence σ_k so that the sets $\sigma_k^{-1}E$ are pairwise disjoint and obtain from (3) the absurd conclusion

$$1 = q_t(\alpha, \mathcal{G}) \geqq \sum_k q_t(\alpha, \sigma_k^{-1}E) = \sum_k q_t(\alpha, E) = \infty.$$

We define the convolution $F * G$ of two measures on $\mathcal{G}$, each of total measure one, by

$$F * G.(E) = \int_{\mathcal{G}} F(d\sigma)G(\sigma^{-1}E) \qquad E \subset \mathcal{G}$$

$$F * G.(\alpha) = F(\alpha) + G(\alpha) - F(\alpha)G(\alpha).$$

With this definition, the semi-group property of the S_t implies $p_s * p_t = p_{s+t}$.

Finally (1) implies at once that $p_t(E) \to 1$ as $t \searrow 0$ whenever E is a neighborhood of e.

Let us call a semi-group satisfying (1), (2) and (3) a probability semi-group. The justification of this name will become clear in Section 5.

Our object is to characterize all probability semi-groups on an arbitrary Lie group $\mathcal{G}$. We deal with the S_t rather than the corresponding $p_t(E)$ because the characterization is most neatly stated in terms of the infinitesimal generator M of the semi-group S_t. Recall that M is the linear transformation whose domain $\mathcal{D}$ is precisely the set of those f in $\bar{\mathcal{C}}$ for which

(4) $$\lim_{t \searrow 0} \frac{1}{t}(S_t f - f)$$

exists, and whose value at an f in $\mathcal{D}$ is given by the limit (4). It is well known that the domain $\mathcal{D}$ of M is dense in $\bar{\mathcal{C}}$ and that the range of each S_t is contained in $\mathcal{D}$. Also, if one defines

$$Rf = \int_0^\infty e^{-t} S_t f \, dt,$$

then R is a one-to-one bounded linear transformation of $\bar{\mathcal{C}}$ on $\mathcal{D}$, and for every f in $\mathcal{D}$

$$R(1 - M)f = f.$$

These properties of the infinitesimal generator all follow from (1) and the fact that the norm of each S_t is one. Taking (2) and (3) into account we proceed to derive the explicit representation of M under the assumption that $\mathcal{G}$ is a Lie group. The next section discusses some preliminaries, two of which — the lemmas in parts D and E — are essential to the main argument. Section 3 proves that the infinitesimal generator of a probability semi-group must have a certain form, and Section 4 proves that every transformation of that form generates a probability semi-group. These results are summarized in the theorem of Section 5; this section also contains a brief interpretation from the point of view of the theory of probability. Finally, in Section 6, we show that when the S_t are 'self-adjoint' the expression for M takes on a particularlysimple form.

We introduce the compactification $\bar{\mathcal{G}}$ (and the attendant nuisance of the exceptional point α) only as a technical device to make several proofs easier than they would be if, say, we treated only functions on $\mathcal{G}$ vanishing at infinity. In order to insure $p_t(\alpha) = 0$, so that the $p_t(E)$ are distributions on $\mathcal{G}$, we have only to require that $\sup_{\tau \varepsilon \mathcal{G}} S_t f.(\tau)$ can be made arbitrarily close to 1 for some function f in $\bar{\mathcal{C}}$ which satisfies $f(\alpha) = 0$ and $f(\tau) \leq 1$ for all τ in $\bar{\mathcal{G}}$.

The consideration of arbitrary Lie groups, rather than separable ones, brings with it no greater generality. One can see this by noting that a finite measure μ on $\bar{\mathcal{G}}$ must vanish on all but a countable number of cosets of the connected component N of e in $\mathcal{G}$. Thus the collection $\mathcal{N}$ of those cosets of N on which $p_t(E)$ differs from zero for some rational t is a countable set; let $\mathcal{G}'$ be the separable group generated by the elements of cosets in $\mathcal{N}$ and $\bar{\mathcal{G}}'$ the compactification of $\mathcal{G}'$ using the same point a as for $\mathcal{G}$. Now consider any function f in $\bar{\mathcal{C}}$ which vanishes outside some compact set which has no point in common with $\bar{\mathcal{G}}'$. Since $\int p_t(d\sigma)f(\sigma)$ vanishes for all rational t and is a continuous function of t, it must vanish for all t. Hence for all t the measure $p_t(E)$ is confined to $\bar{\mathcal{G}}'$. It is easy to see that the semi-group defined by the $p_t(E)$ acting on the continuous functions on $\bar{\mathcal{G}}'$ is a probability semi-group from which the original semi-group S_t can be obtained by left translations.

2. Preliminaries

Let us dispose of some matters which otherwise would interrupt the course of the argument.

A) Let $X^1,\dots,X^n$ be a fixed basis of the left invariant Lie algebra of $\mathcal{G}$ and ν a right invariant Haar measure. Setting $\xi^i_s = \exp(sX^i)$, we have, for every differentiable function f,

$$X^i f_{.}(\tau) = \lim_{s\to 0} \frac{1}{s}\left\{f(\tau\xi^i_s) - f(\tau)\right\}. \qquad (1)$$

It follows that if f and g are smooth and vanish outside a compact set

(2) $$\int_{\mathcal{G}} X^i f_\cdot(\sigma)\, g(\sigma)\, \nu(d\sigma) = - \int_{\mathcal{G}} f(\sigma)\, X^i g_\cdot(\sigma)\, \nu(d\sigma).$$

If Δ is an operator of the form

(3) $$\Delta = \Sigma a_i X^i + \Sigma a_{ij} X^i X^j = \Delta_1 + \Delta_2$$

we may assume that $a_{ij} = a_{ji}$, for $\frac{1}{2}(a_{ij} - a_{ji})(X^i X^j - X^j X^i)$ is an element of the Lie algebra and may be adsorbed into the first sum. The representation of Δ, supposing the a_{ij} symmetric, is unique. It is clear then that any left invariant, second order, linear, differential operator which vanishes on constants may be written in the form (3) with the a_i and a_{ij} constant, and that the operators Δ_1 and Δ_2 are uniquely determined. Making use of (2), we see that

(4) $$\int_{\mathcal{G}} \left\{\Delta_2 f_\cdot(\sigma)\right\} g(\sigma)\, \nu(d\sigma) = \int_{\mathcal{G}} f(\sigma)\, \Delta_2 g_\cdot(\sigma)\, \nu(d\sigma),$$

so that Δ_2 is self-adjoint with respect to the measure ν.

B) Let $\mathcal{C}^k$ be the set of those functions in $\bar{\mathcal{C}}$ which have continuous derivative of order k, and $\bar{\mathcal{C}}^k$ the subset of $\mathcal{C}^k$ comprising those f for which $X^i f, \ldots, X^{i_1} X^{i_2} \ldots X^{i_k} f$ extend by continuity to become functions in $\overset{*}{\mathcal{C}}$.

The norm on $\bar{\mathcal{C}}$ is $\|f\|_0 = \sup_{\tau \in \mathcal{G}} |f(\tau)|$. On $\bar{\mathcal{C}}^k$ we define the norm

$$\|f\|_k = \|f\|_0 + \sum_{i_1, \ldots, i_k} \left\{ \|X^{i_1} f\|_0 + \ldots + \|X^{i_1} X^{i_2} \ldots X^{i_k} f\|_0 \right\},$$

thus making $\bar{\mathcal{C}}^k$ a Banach space. $\bar{\mathcal{C}}^k$, considered as a subspace of $\bar{\mathcal{C}}$, is dense in $\bar{\mathcal{C}}$; the left translations of $\mathcal{G}$ leave $\bar{\mathcal{C}}^k$

invariant; and $||L_\sigma f - f||_k \to 0$ as $\sigma \to e$, provided f is in $\bar{\mathfrak{S}}^k$.

C) In Section 4 we shall use the fact that differential operators can be approximated in the following way by difference quotients.

Consider $\Delta = \Sigma a_i X^i + \Sigma a_{ij} X^i X^j$, where the matrix (a_{ij}) is positive semi-definite. The expression for Δ at e in a suitable coordinate system is

$$\Delta f_{\bullet}(e) = \sum_{1 \leq i \leq n} b_i \frac{\partial f}{\partial x_i}(e) + \sum_{1 \leq i \leq m} \frac{\partial^2 f}{\partial x_i^2}(e)$$

with m some integer between 0 and n. Let us set $K = 1 + \Sigma|b_i|$ and take ρ_r to be the point of $\mathcal{G}$ whose coordinates are $x_j(\rho_r) = b_j/r$, η_r^i the point $x_j(\eta_r^i) = \delta_{ij}/r$, and ζ_r^i the point $x_j(\zeta_r^i) = -\delta_{ij}/r$. For every function f in $\bar{\mathfrak{S}}^2$ the quantity

$$r[f(\rho_r)-f(e)] + r^2 \sum_{1 \leq i \leq m} [f(\eta_r^i)+f(\zeta_r^i)-2f(e)]$$

differs from $\Delta f_{\bullet}(e)$ by terms which are dominated by the oscillations of the first and second derivatives of f, with respect to the coordinates x_i, in the neighborhood $|x_i(\sigma)| < K/r$ of e. Observe that if one considers only sets E which are all contained in a fixed sufficiently small neighborhood of e, then the oscillations of the derivatives $\frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ over E can be majorized by an expression

$$\Sigma c_i \operatorname{osc}_E(X^i f) + \Sigma c_{ij} \operatorname{osc}_E(X^i X^j f)$$

where the coefficients c_i and c_{ij} depend neither on f nor on E.

By making a left translation we have

$$(5) \quad \lim_{r\to\infty} r[f(\tau\rho_r)-f(\tau)] + \lim_{r\to\infty} r^2 \sum_{1\leq i\leq m} [f(\tau\eta_r^i)+f(\tau\xi_r^i)-2f(\tau)] = \Delta f_{\cdot}(\tau)$$

for every f in $\bar{\mathcal{C}}^2$ and every τ in $\mathcal{G}$. It follows from the definition of $\bar{\mathcal{C}}^2$, moreover, that the limits exist uniformly in τ.

Let us set $\Delta f_{\cdot}(\alpha) = 0$. Taking into account the coefficients in (5) we see that there is a sequence of finite positive measures $\mu_r(E)$ on $\mathcal{G} - e$ such that for every f in $\bar{\mathcal{C}}^2$

$$(6) \quad \int_{\mathcal{G}-e} [f(\tau\sigma)-f(\tau)]\, \mu_r(d\sigma) \to \Delta f_{\cdot}(\tau) \qquad r\to\infty$$

in the metric of $\bar{\mathcal{C}}$, provided that $\Delta = \Sigma a_i X^i + \Sigma a_{ij} X^i X^j$ with (a_{ij}) positive semi-definite.

D) For p and f continuous on $\mathcal{G}$, with p vanishing outside some compact set, define

$$p * f_{\cdot}(\tau) = \int_{\mathcal{G}} p(\eta^{-1}) f(\eta\tau)\, \nu(d\eta) = \int_{\mathcal{G}} p(\tau\eta^{-1}) f(\eta)\, \nu(d\eta).$$

If f is in $\bar{\mathcal{C}}^1$ then $p * f$ is in $\bar{\mathcal{C}}^1$ and

$$(7) \quad \begin{aligned} X^i(p * f)_{\cdot}(\tau) &= \lim_{s\to 0} \int p(\eta^{-1}) \frac{f(\eta\tau\xi_s^i) - f(\eta\tau)}{s}\, \nu(d\eta) \\ &= \int p(\eta^{-1}) \lim_{s\to 0} \frac{f(\eta\tau\xi_s^i) - f(\eta\tau)}{s}\, \nu(d\eta) = p * X^i f_{\cdot}(\tau) \end{aligned}$$

because the limit under the integral exists uniformly for $\eta\tau$ restricted to any compact set.

On the other hand, assume that f is in $\bar{\mathcal{C}}$ and that p, besides vanishing outside a compact neighborhood $\mathcal{U}$ of e, belongs to $\bar{\mathcal{C}}^1$. Then $p * f$ belongs to $\bar{\mathcal{C}}^1$ and

$$X^i(p * f)_{\cdot}(\tau) = \lim_{s\to 0} \int \frac{p(\tau\xi_s^i\eta^{-1}) - p(\tau\eta^{-1})}{s} f(\eta)\,\nu(d\eta)$$

$$(8) \qquad = \lim_{s\to 0} \int \frac{p(\tau\xi_s^i\tau^{-1}\eta^{-1}) - p(\eta^{-1})}{s} f(\eta\tau)\,\nu(d\eta)$$

$$= \int \lim_{s\to 0} \frac{p(\tau\xi_s^i\tau^{-1}\eta^{-1}) - p(\eta^{-1})}{s} f(\eta\tau)\,\nu(d\eta)$$

$$= (A_\tau^i p) * f_{\cdot}(\tau)$$

where

$$(9) \qquad A_\tau^i p_{\cdot}(\sigma) = \lim_{s\to 0} \frac{p(\tau\xi_s^i\tau^{-1}\sigma) - p(\sigma)}{s} .$$

A few words must be said about the passage to the limit. Restrict τ to a compact set E and consider only those s which are so small that ξ_s^i belongs to $\mathcal{U}$; then $p(\tau\xi_s^i\tau^{-1}\sigma) - p(\sigma)$ vanishes if σ does not belong to the compact set $E\mathcal{U}^{-1}E^{-1}\mathcal{U}$. The limit in (9) exists uniformly for σ in $E\mathcal{U}^{-1}E^{-1}\mathcal{U}$ and τ in E. So $\frac{1}{s}\left\{p(\tau\xi_s^i\tau^{-1}\sigma) - p(\sigma)\right\}$, considered as a function of σ, is a Cauchy family in $L^1(\nu, \mathcal{G})$ as $s\to 0$ and satisfies the Cauchy condition uniformly for τ in E. Thus the inversion of the integration and limit in (8) is justified.

These are the results we need in proving the following lemma, which will form the basis for the next section.

Lemma. Let $\mathcal{D}$ be a dense linear manifold in $\bar{\mathcal{C}}$ which contains all constants and has the property that $p * g$ is in $\mathcal{D}$ whenever g is in $\mathcal{D}$ and p is a continuous function vanishing outside a compact set, and let the positive number ε and the compact subset E of $\mathcal{G}$ be given. Then (i) for every f in $\mathcal{C}^1$ there is a g in $\mathcal{D}$ such that

$$g \in \mathcal{C}^2,\ g(e) = f(e),\ X^i g_.(e) = X^i f_.(e)$$

$$\|f - g\|_0 < \varepsilon,\ |X^i f_.(\tau) - X^i g_.(\tau)| < \varepsilon \quad \text{for} \quad \tau \in E,$$

and (ii) if f is in $\mathcal{C}^2$ we may further achieve

$$X^i X^j g_.(e) = X^i X^j f_.(e),\ |X^i X^j f_.(\tau) - X^i X^j g_.(\tau)| < \varepsilon \quad \text{for} \quad \tau \in E.$$

We shall content ourselves with the proof of part (i), for the proof of part (ii) is much the same.

Let f then be in $\mathcal{C}^1$. We first choose p in $\mathcal{C}^2$, non-negative, vanishing outside a compact neighborhood $\mathcal{U}$ of e, so that

$$\|p * f - f\|_0 < \frac{\delta}{2}$$

$$|X^i(p*f)_.(\tau) - X^i f_.(\tau)| = |p*(X^i f)_.(\tau) - X^i f_.(\tau)| < \frac{\delta}{2} \text{ for } \tau \in E,$$

and set

$$K = \int p(\cdot\,\eta^{-1})\,\nu(d\eta) + \sum_{1\leq i\leq n}\ \max_{\tau\in E} \int |A^i_\tau p(\eta^{-1})|\,\nu(d\eta).$$

Take $\bar{g}$ in $\mathcal{D}$ so that $\|f - \bar{g}\|_0 < \frac{\delta}{2K}$. Then $\bar{\bar{g}} \equiv p * \bar{g}$ is in $\mathcal{D} \cap \mathcal{C}^2$ and

$$\|p * \bar{g} - p * f\|_0 = \|p * (\bar{g} - f)\|_0 < K \cdot \frac{\delta}{2K} = \frac{\delta}{2}$$

$$|X^i(p*\bar{g})_.(\tau) - X^i(p*f)_.(\tau)| = |(A^i_\tau p)*(\bar{g}-f)_.(\tau)| < K\cdot\frac{\delta}{2K} = \frac{\delta}{2} \text{ for } \tau \in E.$$

So $\|f - \bar{\bar{g}}\|_0 < \delta$ and $|X^i f_.(\tau) - X^i \bar{\bar{g}}_.(\tau)| < \delta$ for $\tau \in E$.

Using what we have just proved, we find functions $\bar{y}_1, \ldots, \bar{y}_n$ in $\mathcal{D}' \cap \mathcal{C}^2$ so that $\bar{y}_i(e)$ and $X^i\bar{y}_{j\cdot}(e)$ are near 0 and δ_{ij}. By a linear change of variables $y_i = \Sigma a_{ij}\bar{y}_j + b_i$ we obtain y_i in $\mathcal{D} \cap \mathcal{C}^2$ satisfying $y_i(e) = 0$, $X^i y_{j\cdot}(e) = \delta_{ij}$. If $A = \sum\limits_i \max\limits_{\tau\varepsilon\bar{\mathcal{G}}} |y_i(\tau)| + \sum\limits_{i,j} \max\limits_{\tau\varepsilon E} |X^i y_{j\cdot}(\tau)|$ then the function

$$g = \bar{\bar{g}} - (\bar{\bar{g}}(e) - f(e)) - \sum_i \left\{X^i\bar{\bar{g}}_{\cdot}(e) - X^i f_{\cdot}(e)\right\} y_i$$

agrees with f at e through terms of first order and also

$$\|f - g\|_0 < \delta + \delta + \delta A = (2 + A)\delta$$
$$|X^i f_{\cdot}(\tau) - X^i g_{\cdot}(\tau)| < \delta + \delta A = (1 + A)\delta \quad \text{for } \tau \in E.$$

Since δ is arbitrary, we have proved part (i).

E) Let $\mathfrak{X}$ be a compact space, $I = [0, \infty)$, and $f = f(t,x)$ a continuous function on $I \times \mathfrak{X}$ with the property:

(10) If $f(t,x_0) = \min\limits_{x\varepsilon\mathfrak{X}} f(t,x)$ then $\frac{\partial f}{\partial t}(t,x_0) \geqq 0$.

It is not difficult to show that then for every $t \geqq 0$

(11) $$\min_{x\varepsilon\mathfrak{X}} f(t,x) \geqq \min_{x\varepsilon\mathfrak{X}} f(0,x)$$

By adding a constant to f we may assume in the proof that $\min f(0,x) = 1$. Let c be positive and consider the function $g(t,x) = e^{ct}f(t,x)$, which achieves its minima (for given t) at the same points of $\mathfrak{X}$ as does $f(t,x)$. Now assume that $g(t,x) < 1 - \varepsilon$ for some pair (t,x) and some ε in $(0,1)$. There must then be a pair (t_0,x_0) such that $g(t_0,x_0) = 1 - \varepsilon$, $t_0 > 0$, and $g(t,x) > 1 - \varepsilon$ for all t in $[0,t_0)$. Consequently $\frac{\partial g}{\partial t}(t_0,x_0) \leqq 0$. On the other hand $g(t_0,x_0) =$

$$\min_{x\varepsilon\mathcal{X}} g(t_o,x) = e^{ct_o} \min_{x\varepsilon\mathcal{X}} f(t_o,x), \text{ so that}$$

$$\frac{\partial g}{\partial t}(t_o,x_o) = e^{ct_o} \frac{\partial f}{\partial t}(t_o,x_o) + cg(t_o,x_o) \geq c(1-\varepsilon) > 0;$$

and we have arrived at a contradiction. Therefore $g(t,x) \geq 1$ for all t and x; by letting c decrease to zero we obtain the same result for f.

Consider now a semi-group $(T_t)_{t>0}$ of bounded linear transformations of the Banach space $\mathcal{C}(\mathcal{X})$ of continuous functions on $\mathcal{X}$ which satisfies

$$\lim_{t\to 0} T_t f = f \qquad f \varepsilon \mathcal{C}(\mathcal{X}),$$

and denote by A the infinitesimal generator of T_t, which we assume to have domain dense in $\mathcal{C}(\mathcal{X})$.

<u>Lemma</u>. The semi-group T_t has the property

(12) $$\min_{x\varepsilon\mathcal{X}} T_t f_{\cdot}(x) \geq \min_{x\varepsilon\mathcal{X}} f(x) \qquad f \varepsilon \mathcal{C}(\mathcal{X})$$

if and only if

(13) $Af_{\cdot}(x_o) \geq 0$ whenever f belongs to the domain of A

and $f(x_o) = \min_{x\varepsilon\mathcal{X}} f(x)$.

Assuming that (12) is true and that Af is defined, we have

$$Af_{\cdot}(x_o) = \lim_{t\searrow 0} \frac{1}{t}\left\{T_t f_{\cdot}(x_o) - f(x_o)\right\}$$

$$\geq \lim_{t\searrow 0} \frac{1}{t}\left\{\min_{x\varepsilon\mathcal{X}} T_t f_{\cdot}(x) - \min_{x\varepsilon\mathcal{X}} f(x)\right\}$$

$$\geq 0.$$

Thus (13) is also true.

Assuming that (13) is true, we note that for f in $\mathcal{C}(\mathcal{X})$ and t positive the function $T_t f$ is in the domain of A, so that

$$\frac{d}{dt} T_t f = A T_t f \qquad t > 0 .$$

Thus the function $f(t,x) = S_t f_{.}(x)$ satisfies (10) and, according to (11), the inequality (12) must hold.

3. <u>The Infinitesimal Generator</u>.

We turn now to the study of the infinitesimal generator M of a probability semi-group S_t on a Lie group $\mathcal{G}$.

If μ is a bounded measure on $\mathcal{G}$ the transformation $L_\mu : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ defined by

$$L_\mu f_{.}(\tau) = \int_{\mathcal{G}} f(\eta\tau)\, \mu(d\eta)$$

is bounded; also $L_\mu S_t = S_t L_\mu$, for S_t commutes with left translations. Hence

$$\lim_{t \searrow 0} \frac{S_t L_\mu f - L_\mu f}{t} = L_\mu \left\{ \lim_{t \to 0} \frac{S_t f - f}{t} \right\}$$

exists for every f in $\mathcal{D}$, so that $L_\mu \mathcal{D}$ is included in $\mathcal{D}$. Moreover $\mathcal{D}$ is dense in $\bar{\mathcal{C}}$ and contains all constants. Consequently the hypotheses of the lemma in part D of Section 2 are fulfilled by $\mathcal{D}$.

An appropriate use of the lemma yields the existence of a function φ in $\mathcal{D} \cap \tilde{\mathcal{C}}^2$ with the properties $\varphi(e) = X^i \varphi_{.}(e) = 0$, $X^i X^j \varphi_{.}(e) = \delta_{ij}$, $\varphi(\tau) > 0$ for τ in $\bar{\mathcal{G}} - e$. In a system $x_1, \ldots, x_n$ of coordinates near e which are fitted to the X^i — that is to say, $x_i(e) = 0$ and $X^i x_{j.}(e) = \delta_{ij}$ — the function

φ behaves like Σx_i^2 near e.

Once φ has been fixed it is an easy consequence of the lemma that for every f in $\mathcal{C}^2$ and every positive ε one can find a function g in $\mathcal{D} \cap \mathcal{C}^2$ such that $|f(\tau) - g(\tau)| \leq \varepsilon\, \varphi(\tau)$ for all τ in $\bar{\mathcal{G}}$.

This being so, we prove that

$$(1)\ Af = \lim_{t \searrow 0} \frac{S_t f\,(e) - f(e)}{t} = \lim_{t \searrow 0} \frac{1}{t} \int_{\bar{\mathcal{G}}} p_t(d\sigma)[f(\sigma) - f(e)]$$

exists for every f in $\mathcal{C}^2$. Given $\varepsilon > 0$, take g in $\mathcal{D}$ so that $|f(\tau) - g(\tau)| \leq \varepsilon\, \varphi(\tau)$. Then

$$\left|\frac{1}{t} \int p_t(d\sigma) f(\sigma) - \frac{1}{t} \int p_t(d\sigma) g(\sigma)\right| \leq \frac{\varepsilon}{t} \int p_t(d\sigma)\, \varphi(\sigma),$$

and the term on the right is $\mathcal{O}(\varepsilon)$ because φ is in $\mathcal{D}$. Consequently, $\frac{1}{t} \int p_t(d\sigma)[f(\sigma) - f(e)]$ differs by $\mathcal{O}(\varepsilon)$ from $\frac{1}{t} \int p_t(d\sigma)[g(\sigma) - g(e)]$; since the latter function tends to a limit as t decreases to zero (since g is in $\mathcal{D}$) and since ε is arbitrary, the limit in (1) must exist.

Consider the family of positive measures

$$F_t(E) = \frac{1}{t} \int_E \varphi(\sigma) p_t(d\sigma)$$

defined on $\bar{\mathcal{G}}$. Since $\lim_{t \to 0} \frac{1}{t} \int \varphi(\sigma) p_t(d\sigma)$ exists the family is uniformly bounded for all t; and according to the relation (1) which we have just proved the integral

$$\int_{\bar{\mathcal{G}} - e} f(\sigma) F_t(d\sigma)$$

converges, as $t \searrow 0$, for every f in $\mathcal{C}^2$ which vanishes at e.

Hence the $F_t(E)$ tend weakly on $\bar{\mathcal{G}} - e$ to a bounded measure $F(E)$. It follows that for a function f in $\bar{\mathcal{C}}$ vanishing at e, we have

$$\frac{1}{t}\int_{\bar{\mathcal{G}}} f(\sigma)\varphi(\sigma)p_t(d\sigma) \to \int_{\bar{\mathcal{G}}-e} f(\sigma)F(d\sigma).$$

Again making use of the lemma in part D of Section 2, we determine elements x_i, x_{ij} of $\mathcal{D}\cap\mathcal{C}^2$ so that $x_i(e) = x_{ij}(e) = X^i x_{jk.}(e) = 0$, $X^i x_{j.}(e) = \delta_{ij}$, and $X^i X^j x_{kl}(e)$ are zero unless the pair i,j is the same as the pair k,l — in which case the value is 1 if $i \neq j$ and 2 if $i = j$. Thus the x_i serve as a coordinate system at e and the x_{ij} behave like the products $x_i x_j$ through terms of second order at e.

Take any f in $\mathcal{C}^2$ and set $f(e) = c$, $X^i f_.(e) = c_i$, $X^i X^j(f - \Sigma c_k x_k)_.(e) = c_{ij}$. Then $f - c - \Sigma c_i x_i - \frac{1}{2}\Sigma c_{ij}x_{ij}$ is in $\mathcal{C}^2$ and has the form $h\varphi$ with h in $\bar{\mathcal{C}}$ and $h(e) = 0$. Hence, setting

$$b_i = \lim_{t \searrow 0}\frac{1}{t}\int_{\bar{\mathcal{G}}} p_t(d\sigma)x_i(\sigma), \quad b_{ij} = \frac{1}{2}\lim_{t\searrow 0}\frac{1}{t}\int_{\bar{\mathcal{G}}} p_t(d\sigma)x_{ij}(\sigma)$$

we have

$$\lim_{t\searrow 0}\frac{S_t f_.(e) - f(e)}{t} = \lim_{t\searrow 0}\frac{S_t(\Sigma c_k x_k + \frac{1}{2}\Sigma c_{kl}x_{kl})_.(e)}{t} + \lim_{t\searrow 0}\frac{S_t(h\varphi)_.(e)}{t}$$

$$= \Sigma b_i c_i + \Sigma b_{ij}c_{ij} + \int_{\bar{\mathcal{G}}-e}\frac{f(\sigma) - c - \Sigma c_i x_i(\sigma) - \frac{1}{2}\Sigma c_{ij}x_{ij}(\sigma)}{\varphi(\sigma)} F(d\sigma),$$

where we have written out h explicitly in the last integral.

Note that the integral $\int_{\bar{\mathcal{G}}-e} x_{ij}/\varphi F(d\sigma)$ makes sense, because x_{ij}/φ is a bounded continuous function on $\bar{\mathcal{G}} - e$. Thus we may drop the term $\frac{1}{2}\int \Sigma c_{ij}x_{ij}/\varphi F(d\sigma)$ from the integral if we agree to adsorb it in the sum $\Sigma b_{ij}c_{ij}$ by changing the values of the b_{ij} appropriately. After this change is made we can write

$$(2)\ Af = \Sigma a_i X^i f_{\cdot}(e) + \Sigma a_{ij} X^i X^j f_{\cdot}(e) + \int_{\bar{\mathcal{G}}-e} \left\{ f(\sigma) - f(e) - \Sigma X^i f_{\cdot}(e)\, x_i(\sigma) \right\} G(d\sigma)$$

$$= \Delta_1 f_{\cdot}(e) + \Delta_2 f_{\cdot}(e) + \int_{\bar{\mathcal{G}}-e} \left\{ f(\sigma) - f(e) - \Sigma X^i f_{\cdot}(e) x_i(\sigma) \right\} G(d\sigma)$$

in just one way with symmetric a_{ij} (see part A of Section 2), and $G(E)$ is the positive measure

$$G(E) = \int_E \frac{1}{\varphi(\sigma)} F(d\sigma), \qquad E \subset \bar{\mathcal{G}} - e$$

Here the measure $G(E)$ and the operator Δ_2 are uniquely determined by A, whereas Δ_1 is determined only after the x_i have been chosen. To see this note first that for f in $\mathcal{C}_0^2$ and vanishing near e

$$Af = \int_{\bar{\mathcal{G}}-e} f(\sigma) G(d\sigma),$$

which proves that $G(E)$ is uniquely defined. That being so, the part of A not accounted for by the integral is also well defined once the x_i are chosen; but changing the x_i changes only Δ_1, so Δ_2 is uniquely defined by A.

It is easy to verify that Δ_2 is elliptic — that is to say, the form $\Sigma a_{ij} \lambda_i \lambda_j$ is positive semi-definite. To see this we confine our attention to those f in $\mathcal{D} \cap \mathcal{C}^2$ for which

$f(e) = X^i f_{\cdot}(e) = 0$ and $f(\tau) \geq 0$ for $\tau \neq e$. Since $f(e) = \min_{\sigma \varepsilon \bar{\mathcal{G}}} f(\sigma)$ and $S_t f_{\cdot}(\tau) \geq 0$ for all $\tau \varepsilon \bar{\mathcal{G}}$, we have

$$(4) \qquad Af = \lim \frac{1}{t} S_t f_{\cdot}(e) = \Delta_2 f_{\cdot}(e) + \int_{\bar{\mathcal{G}}-e} f(\sigma)\, G(d\sigma) \geq 0.$$

Assume, for the sake of arguing by reductio ad absurdum, that $\Delta_2 f_{\cdot}(e)$ is negative for some such f. Making use of the lemma in part D of Section 2 we construct a sequence of functions f_k, of the same type as f, such that $X^i X^j f_{k\cdot}(e) = X^i X^j f_{\cdot}(e)$ and

$$\int_{\bar{\mathcal{G}}-e} f_k(\sigma)\, G(d\sigma) \to 0 \quad \text{as} \quad k \to \infty.$$

Thus for large k we have $Af_k < 0$, which contradicts (4). Now, the matrices $(X^i X^j f_{\cdot}(e))$ range over all positive semi-definite matrices. Since $\Delta_2 f_{\cdot}(e)$ is always non-negative the form $\Sigma a_{ij} \lambda_i \lambda_j$ must therefore be positive semi-definite.

When demonstrating in the next section that $\mathcal{D}$ always includes $\bar{\mathcal{C}}^2$ we must know beforehand that the linear functional A on $\bar{\mathcal{C}}^2$ completely determines the probability semi-group S_t from which it is derived. To see this note first that A on $\mathcal{C}^2$ determines the constants a_i and a_{ij} and the measure $G(E)$, so that A is determined on all of $\bar{\mathcal{C}}^2$. Now consider any f in $\bar{\mathcal{C}}^2$ which is constant outside a compact set not containing α; since such functions are dense in $\bar{\mathcal{C}}$ it will suffice to prove that A on $\bar{\mathcal{C}}^2$ determines $S_t f$ for all positive t. It follows from the discussion of $p_t(\alpha, E)$ that $S_t f_{\cdot}(\alpha)$ has the constant value $f(\alpha)$. Also $S_t f$ belongs to $\bar{\mathcal{C}}^2$, because the derivatives of f vanish outside a compact set. Hence, setting $f_o(\tau) = f(\tau)$ and $f_t(\tau) = S_t f_.(\tau)$ for positive t, we obtain a continuous function $f_t(\tau)$ on $[0,\infty) \times \bar{\mathcal{G}}$ which satisfies the equations

$$\frac{\partial f_t(a)}{\partial t} = 0$$

$$\frac{\partial f_t(\tau)}{\partial t} = \lim_{s \searrow 0} \frac{1}{s} \left\{ S_s S_t f_.(\tau) - S_t f_.(\tau) \right. \qquad \tau \in \mathcal{G}$$

$$= A(L_\tau S_t f)$$

$$= \Delta_1 f_t(\tau) + \Delta_2 f_t(\tau) + \int_{\bar{\mathcal{G}}-e} \left\{ f_t(\tau\sigma) - f_t(\tau) - \Sigma X^i f_t(\tau) x_i(\sigma) \right\} G(d\sigma).$$

The fact that $S_t f$ is uniquely determined by A on $\mathcal{C}^2$ is therefore equivalent to the following statement, which is but a particular instance of the preliminary result in part E of Section 2 because Δ_2 is elliptic: Let $g_t(\tau)$ be a continuous function on $[0,\infty) \times \bar{\mathcal{G}}$ such that $g_o(\tau) = 0$ for all τ in $\bar{\mathcal{G}}$, $g_t(a) = 0$ for all positive t, $\frac{\partial g_t(\tau)}{\partial t}$ exists for all positive t and all τ in $\bar{\mathcal{G}}$, $g_t(\tau)$ is twice differentiable on $\mathcal{G}$ for each positive t. If also

$$\frac{\partial g_t(\tau)}{\partial t} = \Delta_1 g_t(\tau) + \Delta_2 g_t(\tau) + \int_{\bar{\mathcal{G}}-e} \left\{ g_t(\tau\sigma) - g_t(\tau) - \Sigma X^i g_t(\tau)\, x_i(\sigma) \right\} G(d\sigma)$$

then $g_t(\tau)$ must vanish identically. This completes the proof that A on $\bar{\mathcal{C}}^2$ determines the semi-group S_t.

4. The Generation of Probability Semi-Groups

The preceding section shows that a probability semi-group defines on $\bar{\mathcal{C}}^2$ a linear functional which uniquely determines the semi-group. Turning matters around, we now prove that every linear functional on $\bar{\mathcal{C}}^2$ satisfying the obvious necessary conditions gives rise to a probability semi-group.

Let the linear functional A be defined on $\bar{\mathcal{C}}^2$ by

$$(1)\quad Af = \Sigma a_i X^i f_{\cdot}(e) + \Sigma a_{ij} X^i X^j f_{\cdot}(e) + \int_{\bar{\mathcal{G}}-e} \left\{ f(\sigma) - f(e) - \Sigma X^i f_{\cdot}(e) x_i(\sigma) \right\} G(d\sigma)$$

where the a_i, a_{ij}, x_i, and G satisfy the following conditions

(a) $x_1, \ldots, x_n \in \bar{\mathcal{C}}^2$, $x_i(e) = 0$, $X^i x_{j\cdot}(e) = \delta_{ij}$

(b) $a_{ij} = a_{ji}$, $\Sigma a_{ij} \lambda_i \lambda_j \geqq 0$ for all λ_i

(c) $G(E) = \int_E \frac{1}{\varphi(\sigma)} F(d\sigma)$, where F is a bounded positive measure on $\bar{\mathcal{G}} - e$ and φ is a function in $\bar{\mathcal{C}}^2$ with $\varphi(e) = X^i \varphi_{\cdot}(e) = 0$, $X^i X^j \varphi_{\cdot}(e) = \delta_{ij}$, $\varphi(\tau) > 0$ for $\tau \in \bar{\mathcal{G}} - e$.

It is clear that (1) then defines a bounded linear functional on the Banach space $\bar{\mathcal{C}}^2$ and that the equations

$$(2)\quad \begin{aligned} Nf_{\cdot}(\tau) &= A(L_\tau f) \qquad \tau \in \mathcal{G} \\ Nf_{\cdot}(a) &= \lim_{\tau \to a} A(L_\tau f) = 0 \end{aligned}$$

define a bounded linear transformation from $\bar{\mathcal{C}}^2$ to $\bar{\mathcal{C}}$. We are going to show that N is the restriction to $\bar{\mathcal{C}}^2$ of the infinitesimal generator of exactly one probability semi-group.

First let us consider an A which can be written

$$(3)\quad Af = \int_{\bar{\mathcal{G}}-e} \left\{ f(\sigma) - f(e) \right\} G(d\sigma)$$

with G a bounded positive measure on $\bar{\mathcal{G}} - e$; such an A is obviously a special case of (1). Then N is given by the expression

$$(4) \qquad Nf.(\tau) = \int_{\bar{\mathcal{G}}-e} \{f(\tau\sigma) - f(\tau)\}\, G(d\sigma) \qquad \tau \in \mathcal{G}$$

and this equation defines N as a linear transformation from the submanifold $\bar{\mathcal{C}}^2$ of $\bar{\mathcal{C}}$ to $\bar{\mathcal{C}}$ itself which has bound $\|N\|_0 = 2G(\bar{\mathcal{G}} - e)$ when $\|f\|_0$ is taken as the norm on $\bar{\mathcal{C}}^2$. Consequently N extends by continuity to become a linear transformation $\bar{N}: \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ with the bound $2G(\bar{\mathcal{G}} - e)$ and the same expression (4) for all $f \in \bar{\mathcal{C}}$.

It follows that the $S_t \equiv \exp(t\bar{N})$ for $t \geqq 0$ form a semi-group of bounded linear transformations on $\bar{\mathcal{C}}$ which is continuous in the norm topology of operators. Since $\bar{N}f.(\tau_0) \geqq 0$ whenever $f(\tau_0) = \min_{\tau \in \bar{\mathcal{G}}} f(\tau)$ the lemma in part E of Section 2 assures us that $\min S_t f \geqq \min f$ for $t > 0$ and $f \in \bar{\mathcal{C}}$. Finally, S_t commutes with left translation of $\mathcal{G}$ because $\bar{N}$ does so.

Thus whenever A has the form (3) the corresponding N is the restriction to $\bar{\mathcal{C}}^2$ of the infinitesimal generator of a probability semi-group.

We settle the general case by approximating. First, however, we must establish a number of incidental results.

The measure $G(E)$ occuring in (1) is such that the integral $\int_{\bar{\mathcal{G}}-e} \varphi(\sigma)\, G(d\sigma)$ is finite; by the same reasoning as in Section 1 it follows that we may assume $\mathcal{G}$ to be separable. In this

We use these facts in proving a critical lemma.

Lemma. Let M_k be the infinitesimal generator of a probability semi-group $S_t^{(k)}$. Suppose that M_k is defined on $\bar{\mathcal{C}}^2$ and that as $k \to \infty$ the function $M_k f$ converges in the norm of $\bar{\mathcal{C}}$, say to $\bar{M}f$, for every f in $\bar{\mathcal{C}}^2$. Then $S_t^{(k)}f$ converges, say to $S_t f$, for every f in $\bar{\mathcal{C}}$ and $t > 0$; the transformations S_t form a probability semi-group; and the infinitesimal generator of S_t is defined on $\bar{\mathcal{C}}^2$ and coincides there with $\bar{M}$.

First we have

$$\left\|\frac{dS_t^{(k)}f}{dt}\right\|_o = \|S_t^{(k)}M_k f\|_o \leq \|M_k f\|_o$$

for f in $\bar{\mathcal{C}}^2$; so the $S_t^{(k)}f$, considered as functions of t, have a common modulus of continuity, provided f is in $\bar{\mathcal{C}}^2$. The result preceding the lemma may now be applied to obtain a subsequence, which we again denote by $S_t^{(k)}$, and a probability semi-group S_t having the property that $S_t^{(k)}f \to S_t f$ for every $f \in \bar{\mathcal{C}}$.

We recall that in the theory of semi-groups one proves that the linear transformation

$$R^{(k)}f = \int_0^\infty e^{-t}S_t^{(k)}f\,dt \tag{5}$$

is a one-to-one transformation whose bound on $\bar{\mathcal{C}}$ is 1 and which maps $\bar{\mathcal{C}}$ onto the domain $\mathcal{D}_k$ of M_k. Also

$$R^{(k)}f - R^{(k)}M_k f = f \qquad \text{for } f \in \mathcal{D}_k. \tag{6}$$

Let us pass to the limit here. We define R by an equation similar to (5) and denote by M the infinitesimal generator of S_t. Taking an f in $\bar{\mathcal{C}}^2$ and letting $k \to \infty$, equation (6) becomes

$$Rf - R\bar{M}f = f \qquad f \in \bar{\mathcal{C}}^2.$$

This equation tells us first that f is in the range of R, hence in the domain of M; next, that $\bar{M}f = Mf$, for Mf satisfies the same equation as $\bar{M}f$ and R is one-to-one. This proves the lemma for the subsequence. But note that according to the result at the end of Section 2 $\bar{M}$ determines S_t; so our construction of S_t does not depend on the subsequence chosen. A familiar argument then shows that the full original sequence of semi-groups must converge to S_t.

We are now in a position to show that a linear functional given by (1), with conditions (a)(b)(c) holding, defines by equations (2) the restriction N to $\bar{\mathcal{C}}^2$ of the infinitesimal generator of a probability semi-group.

Consider an A of the form

$$\text{(7)} \qquad Af = \int_{\bar{\mathcal{G}}-e} \left\{ f(\sigma) - f(e) - \Sigma X^i f(e).x_i(\sigma) \right\} G(d\sigma)$$

with G bounded on $\bar{\mathcal{G}} - e$. We may write

$$Af = Xf.(e) + \int_{\bar{\mathcal{G}}-e} \left\{ f(\sigma) - f(e) \right\} G(d\sigma)$$

where $X = -\Sigma_i \left(\int x_i(\sigma)\, G(d\sigma) \right).X^i$ is an element of the Lie algebra of $\mathcal{G}$. We have seen in part C of Section 2 that there is a sequence of bounded positive measures μ_k on $\bar{\mathcal{G}} - e$ such that $\int \left\{ f(\tau\sigma) - f(\tau) \right\} \mu_k(d\sigma) \to Xf$ in $\mathcal{C}$ whenever f

belongs to $\bar{\mathcal{C}}^2$. Now, by what has already been proved, the transformations

$$N_k f.(\tau) = \int_{\bar{\mathcal{G}}-e} \left\{f(\tau\sigma) - f(\tau)\right\} \left\{G(d\sigma) + \mu_k(d\sigma)\right\}$$

are the generators of probability semi-groups. Hence, according to the lemma, the N defined by A is the restriction to $\bar{\mathcal{C}}^2$ of the infinitesimal generator of a probability semi-group.

The same argument disposes of an A of the form

$$Af = \Sigma a_i X_i + \int_{\bar{\mathcal{G}}-e} \left\{f(\sigma) - f(e) - \Sigma X^i f.(e) x_i(\sigma)\right\} G(d\sigma)$$

when G is bounded on $\bar{\mathcal{G}} - e$.

Consider now the general case. Take a sequence of bounded positive measures μ_k on $\bar{\mathcal{G}} - e$ so that $\int \left\{f(\tau\sigma) - f(\tau)\right\} \mu_k(d\sigma)$ tends in $\bar{\mathcal{C}}$ to $\Sigma a_{ij} X^i X^j f$ for every f in $\bar{\mathcal{C}}^2$ and let $G_k(E)$ be the bounded measure on $\bar{\mathcal{G}} - e$ defined by $G_k(E) = G(E \cap (\bar{\mathcal{G}} - \mathcal{U}_k))$ where $\mathcal{U}_k$ is a decreasing sequence of compact neighborhoods of e shrinking down to e. If we define

$$N_k f.(\tau) = \Sigma a_i X_i f.(\tau) + \int_{\bar{\mathcal{G}}-e} (f(\tau\sigma) - f(\tau))\, \mu_k(d\sigma) +$$
$$\int_{\bar{\mathcal{G}}-e} \left\{f(\tau\sigma) - f(\tau) - \Sigma X^i f.(\tau) x_i(\sigma)\right\} G_k(d\sigma)$$

then N_k coincides on $\bar{\mathcal{C}}^2$ with the infinitesimal generator of a probability semi-group and also $N_k f \to Nf$ in $\bar{\mathcal{C}}$ for every f in $\bar{\mathcal{C}}^2$. Hence N is the restriction to $\bar{\mathcal{C}}^2$ of the infinitesimal generator of a probability semi-group.

At this point we can verify that the infinitesimal generator of a probability semi-group is always defined on $\bar{\mathcal{C}}^2$. Let S_t be the semi-group and A the linear functional on $\bar{\mathcal{C}}^2$ derived from the S_t as in Section 3. When this A is taken as the A of equation (1), the equations (2) define the restriction N to $\bar{\mathcal{C}}^2$ of the infinitesimal generator of some semi-group, say $\bar{S}_t$. But, according to the last result of Section 3, the two semi-groups S_t and $\bar{S}_t$ must coincide, for they induce the same linear functional A on $\bar{\mathcal{C}}^2$.

5. Statement of Results

Collecting what we have proved in the preceding sections we have:

Theorem 1. Let S_t be a probability semi-group on a Lie group $\mathcal{G}$. The infinitesimal generator M of S_t is defined at least on $\bar{\mathcal{C}}^2$ and has there the expression

(1) $$Mf.(\tau) = \Sigma a_i X^i f.(\tau) + \Sigma a_{ij} X^i X^j f.(\tau) + \int_{\bar{\mathcal{G}}-e} \left\{ f(\tau\sigma) - f(\tau) - \Sigma X^i f.(\tau) x_i(\sigma) \right\} G(d\sigma),$$

where

(a) $x_1, \ldots, x_n \in \bar{\mathcal{C}}^2$, $x_i(e) = 0$, $X^i x_j.(e) = \delta_{ij}$

(b) $a_{ij} = a_{ji}$, $\Sigma a_{ij} \lambda_i \lambda_j \geq 0$ for all λ_i

(c) $G(E) = \int_E \frac{1}{\varphi(\sigma)} F(d\sigma)$, where F is a bounded positive measure on $\bar{\mathcal{G}} - e$ and φ is in $\bar{\mathcal{C}}^2$ with $\varphi(e) = X^i \varphi.(e) = 0$, $X^i X^j \varphi.(e) = \delta_{ij}$, $\varphi(\tau) > 0$ for $\tau \in \bar{\mathcal{G}} - e$.

The measure G and the operator $\Sigma a_{ij} X^i X^j$ are determined by S_t independently of the choice of the X^i and x_i. Moreover the restriction of M to $\bar{\mathcal{C}}^2$ determines the semi-group S_t.

Conversely, if the operator $M: \bar{\mathcal{C}}^2 \to \bar{\mathcal{C}}$ is defined by (1) and if conditions (a)(b)(c) are satisfied, then M is the restriction to $\bar{\mathcal{C}}^2$ of the infinitesimal generator of exactly one probability semi-group.

We add a few remarks.

First, $G(\alpha) = 0$ if $p_t(\alpha) = 0$ for some $t > 0$; conversely, $p_t(\alpha) = 0$ for all $t > 0$ if $G(\alpha) = 0$.

Second, M is a bounded operator on $\bar{\mathcal{C}}$ if and only if

$$(2) \qquad Mf.(\tau) = \int_{\bar{\mathcal{G}}-e} [f(\tau\sigma) - f(\tau)]\, F(d\sigma)$$

with F a positive bounded measure on $\bar{\mathcal{G}} - e$.

Third, if $\mathcal{G}$ is discrete then M is necessarily of the form (2).

Fourth, the set of generators of probability semi-groups is a positive cone in which operators of the form (2) are dense. To be precise, let $\mathcal{M}$ be the set of restrictions to $\bar{\mathcal{C}}^2$ of infinitesimal generators of probability semi-groups. Then $\mathcal{M}$ is closed under the operation of forming linear combinations with positive coefficients; furthermore $\mathcal{M}$ is closed in the strong topology of linear operators from the Banach space $\bar{\mathcal{C}}^2$ to $\bar{\mathcal{C}}$, and the operators (2) are dense in $\mathcal{M}$ in this topology (see the Lemma of Section 4 and the subsequent argument).

Finally, we sketch the probability interpretation of our results. The definition $q_t(\sigma, E) = p_t(\sigma^{-1}E)$ yields the transition probabilities of a stationary Markoff process on $\mathcal{G}$ (assuming that $p_t(a) = 0$) which is the invariant under left translations, $q_t(\tau\sigma, \tau E) = q_t(\sigma, E)$. By choosing an appropriate probability space one may 'realize' the transition probabilities by a stationary process of random variables X_t with values in $\mathcal{G}$ and independent increments. This is to say,

$$Pr\left\{X_s^{-1}X_t \ \varepsilon\ E\right\} = p_{t-s}(E) \qquad 0 < s < t$$

and the increments $X_{t_1}^{-1}X_{t_2}, X_{t_2}^{-1}X_{t_3}, \ldots, X_{t_{k-1}}^{-1}X_{t_k}$ are independent if $0 < t_1 < \ldots < t_k$.

If M has the simple form $\lambda(L_\sigma - 1)$ the process X_t becomes a Poisson process in this sense: If the cyclic group $\{\sigma^k\}$ is infinite an easy calculation shows that

$$S_t f.(\tau) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} f(\tau\sigma^k),$$

and X_t has the distribution $Pr\left\{X_t = \sigma^k\right\} = e^{-\lambda t}(\lambda t)^k/k!$ or 0 according as $k \geq 0$ or $k < 0$. If $\{\sigma^k\}$ has finite order r then

$$S_t f.(\tau) = \sum_{k=0}^{r-1} K_k(\lambda t)\ f(\lambda\sigma^k)$$

where $K_k(t) = e^{-t} \Sigma\ t^l/l!$, the sum being extended over $l \equiv k$ mod r, $l \geq 0$.

For an M of the form (2) we have

$$p_t(e) = 1 - t\ F(\mathcal{G} - e) + o(t) \quad t \to 0$$

$$p_t(E) = t\ F(E) + o(t) \quad E \subset \mathcal{G} - e,\ t \to 0.$$

Thus $\Pr\left\{X_s^{-1}X_t = e\right\} \sim 1 - (t - s)\, F(\mathcal{G} - e)$ and $\Pr\left\{X_s^{-1}X_t \in E\right\} \sim (t - s)\, F(E)$ for $t - s$ small and $E \subset \bar{\mathcal{G}} - e$.

If $M = \Delta_1$ then S_t is just right multiplication by the group element $\exp(t\Delta_1)$. The processes for which $M = \Delta_1 + \Delta_2$ are the 'generalized' Gaussian processes; a detailed construction of such processes may be found in [1].

6. Self-Adjoint Probability Semi-Groups.

When the $p_t(E)$ corresponding to S_t are symmetric in the sense that $p_t(E) = p_t(JE)$, where J is the involution $\sigma \to \sigma^{-1}$ extended to $\bar{\mathcal{G}}$ by setting $J(\alpha) = \alpha$, the infinitesimal generator M — or, what amounts to the same thing, the functional A derived from M -- can be written in a particularly simple form. It follows from the invariance of $p_t(E)$ that $G(JE) = G(E)$, so that the integral

$$\int_{\mathcal{G}-e} [f(\sigma) - f(e) - \Sigma X^i f.(e) x_i(\sigma)]\, G(d\sigma)$$

may be written

$$\frac{1}{2} \int_{\mathcal{G}-e} [f(\sigma) + f(\sigma^{-1}) - 2f(e) - \Sigma X^i f.(e)\left\{x_i(\sigma) + x_i(\sigma^{-1})\right\}]G(d\sigma).$$

Here the integrals $\int [x_i(\sigma) + x_i(\sigma^{-1})]\, G(d\sigma)$ exist in the ordinary sense because $x_i(\sigma) + x_i(\sigma^{-1})$ is of second order at e. This being so we may write, adsorbing the last terms under the integral into Δ_1,

$$(1)\quad Af = \Delta_1 f.(e) + \Delta_2 f.(e) + \frac{1}{2}\int_{\mathcal{G}-e} [f(\sigma) + f(\sigma^{-1}) - 2f(e)]\, G(d\sigma) + G(\alpha)[f(\alpha) - f(e)].$$

It is clear that A must take on the same values for f and $\bar{f} \equiv f \circ J$. In view of the facts that $\Delta_1 \bar{f}.(e) = -\Delta_1 f.(e)$, that Δ_2 is a second order operator, and that the last two terms of (1) remain unchanged when f is replaced by $\bar{f}$, — we must have $\Delta_1 = 0$. Then Δ_2 must also satisfy the condition $\Delta_2 \bar{f}.(e) = \Delta_2 f.(e)$. So we may say: If the $p_t(E)$ satisfy $p_t(JE) = p_t(E)$ then the linear functional A derived from the semi-group S_t may be written

$$(2)\ Af = \Delta_2 f.(e) + \frac{1}{2}\int_{\mathcal{G}-e} [f(\sigma) + f(\sigma^{-1}) - 2f(e)]\, G(d\sigma) + G(a)[f(a) - f(e)]$$

where the functional $f \to \Delta_2 f.(e)$ and the measure G are invariant under J. Moreover M and S_t are completely determined by the restriction of A to the class of functions f in $\bar{\mathcal{C}}^2$ satisfying $f(\sigma) = f(\sigma^{-1})$.

The symmetry we have just discussed implies many interesting properties, both probabilistic and analytical, of the semi-group S_t. In the simplest case, that of discrete $\mathcal{G}$ and $p_t(a) = 0$, it is equivalent to symmetry of the transition probabilities

$$q_t(\sigma, \tau) = p_t(\sigma^{-1}\tau) = p_t(\ ^{-1}\sigma) = q_t(\tau, \sigma).$$

In the general situation the symmetry expresses itself most significantly in terms of the Markoff transition probabilities inverse to the ones introduced in the preceding section. To carry out this interpretation in detail would, however, lead us to far astray; instead, we develop one of the analytical consequences of the symmetry.

Let S_t be a probability semi-group whose associated measures $p_t(E)$ are confined to $\mathcal{G}$ itself, and let L^2 be the Hilbert space of functions on $\mathcal{G}$ which are square integrable with respect to the right invariant Haar measure ν. The formula

$$T_t f.(\tau) = \int_{\mathcal{G}} f(\tau\sigma)\, p_t(d\sigma) \qquad f \in L^2$$

then defines T_t as a linear transformation, with bound one, of L^2 into itself; and the T_t clearly form a semi-group satisfying $\lim T_t f = f$ for every f in L^2, so that the infinitesimal generator N is defined on a dense subset of L^2. It is easy to see that N coincides with the infinitesimal generator M of the semi-group S_t on the set of those functions f in $\mathcal{F}^2$ for which f, $X^i f$, $X^i X^j f$ all belong to L^2 (and hence vanish at infinity).

We are going to find necessary and sufficient conditions for the T_t to be normal or self-adjoint operators. Since

$$\int_{\mathcal{G}} \overline{g(\tau)}\, \nu(d\tau) \int_{\mathcal{G}} f(\tau\sigma) p_t(d\sigma) = \int_{\mathcal{G}} f(\tau) \nu(d\tau) \int_{\mathcal{G}} \overline{g(\tau\sigma^{-1})} p_t(d\sigma)$$

the adjoint of T_t is

$$T_t^* f.(\tau) = \int_{\mathcal{G}} f(\tau\sigma) p_t'(d\sigma)$$

where $p_t'(E) = p_t(JE)$, a notation we shall employ later on for other measures too. A simple calculation shows that T_t is normal if and only if $p_t * p_t' = p_t' * p_t$ and that T_t is self-adjoint if and only if $p_t' = p_t$.

Let us assume that all T_t are normal and draw some conclusions about N. Zero is never a characteristic value of T_t, because $T_t f \to f$ as $t \to 0$; consequently we have the representation

$$T_t f = \int e^{tz} E(dz) f$$

where the complex spectral measure E is confined to the half plane $\mathcal{R} z \leqq 0$ because each T_t has bound one. It follows that the infinitesimal generator N is the normal transformation $\int z E(dz) f$, defined for every f in L^2 for which the integral $\int |z|^2 (E(dz)f, f)$ is finite. One can verify this in the following way: If $\int |z|^2 (E(dz)f,f)$ is finite then

$$\left\| \frac{1}{t}(T_t f - f) - \int z E(dz) f \right\|^2 = \int \frac{1}{t^2} |e^{tz} - 1 - tz|^2 (E(dz)f,f)$$

and the last integral tends to zero with t, because the integrand tends to zero and is less than $|z|^2$ for $\mathcal{R} z \leqq 0$. Conversely, if $\frac{1}{t}(T_t f - f)$ converges in L^2 as t decreases to zero, then for some constant K

$$\begin{aligned} K &\geqq \limsup_{t \searrow 0} \left\| \frac{1}{t}(T_t f - f) \right\|^2 \\ &= \limsup_{t \searrow 0} \int \frac{1}{t^2} |e^{tz} - 1|^2 (E(dz)f,f) \\ &\geqq \int \lim_{t \searrow 0} \frac{1}{t^2} |e^{tz} - 1|^2 (E(dz)f,f) \\ &= \int |z|^2 (E(dz)f,f), \end{aligned}$$

so that f belongs to the domain of the transformation $\int z E(dz) f$ and the preceding argument applies.

We translate now the conditions on the $p_t(E)$ into conditions on the generator M by means of two remarks, whose proof we leave to the reader.

First, consider two probability semi-groups S_t^1 and S_t^2 with generators M^1 and M^2. For S_s^1 and S_t^2 to commute for all positive s and t it is necessary and sufficient that $M^1M^2f = M^2M^1f$ for all f belonging to $\bar{\mathcal{C}}^4$.

Second, let S_t be a probability semi-group with generator M, associated measures $p_t(E)$, and associated linear functional A. Then $A'f \equiv A(f \circ J)$ is obviously a linear functional which defines the generator M' of a probability semi-group S_t'. The measures associated with S_t' are precisely the $p_t'(E) \equiv p_t(JE)$.

From these remarks it follows that the statements $p_t * p_t' = p_t' * p_t$ and $p_t' = p_t$ are equivalent respectively to $S_tS_t' = S_t'S_t$ and $S_t' = S_t$ and hence to $MM' = M'M$ and $M = M'$.

Let us consider the special case in which

$$Af = \int_{\mathcal{G}-e} [f(\sigma) - f(e)]\, G(d\sigma)$$

with G a bounded measure on $\mathcal{G} - e$. Then T_t is normal if and only if $G * G' = G' * G$. The necessity follows at once from the fact that G is the weak limit of the measures $\frac{1}{t}p_t(E)$ and the sufficiency is proved by noting that the infinitesimal generator N of the T_t is defined on all of L^2 by the formula

$$Nf.(\tau) = \int_{\mathcal{G}-e} [f(\tau\sigma) - f(\tau)]\, G(d\sigma),$$

and using the same reasoning as for the transformations T_t. It is hardly worthwhile to write down the commutativity of M and M' in the general case in terms of the three components of M, for the relations are too cumbersome to be of use.

The condition $M = M'$ is equivalent, as we have seen, to the fact that A can be written in the special form (2), with of course $G(\alpha) = 0$ if we are dealing with the transformations T_t.

We have proved incidentally that Δ_1 and Δ_2 occuring in the resolution of the infinitesimal generator N are always normal and that Δ_2 is even self adjoint.

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2) This paper grew out of conversations with Salomon Bochner, whose contributions to the general subject are too numerous to cite. It was he who suggested that the last section be included.

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Lévy and Khinchin have given an explicit representation of the characteristic function of an infinitely divisible probability distribution on the additive group of the real numbers. Their results are extended here to semi-groups of probability distributions on arbitrary Lie groups. The rôle of the characteristic function is taken over by the infinitesimal generator of a semi-group of transformations associated with the probability distributions.

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